Delinquency and Default of Securitized Hotel Mortgages*

Daniel C. Quan
442 Statler Hall
School of Hotel Administration
Cornell University
Ithaca, NY 14852
(607) 255-6404
dq22@cornell.edu

Daniel Lebret
School of Hotel Administration
Cornell University
dl255@cornell.edu

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Abstract

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We specify and estimate a structural model of securitized hotel mortgage delinquency and default. Using a unique loan-level panel database, we are able to distinguish between various stages of loan distress prior to the default state. We view arriving at these states as the outcome of a dynamic programming problem solved by the borrower thus our structural model captures the decision making process of the borrower at the loan level. This structural approach can potentially yield a better predictive model of the default and delinquency decision even in economic environments the CMBS market has yet to experience such as during an downturn in commercial real estate prices. We are also able to incorporate contemporaneous property level performance measures from our access to competitive set information for each hotel from Smith Travel Research. Another innovation of our paper is the use of a cumulative wealth variable which captures the performance of the property in prior periods to explain the subsequent delinquency or default decisions. We find that despite the increased complexity of our model, reasonable results were obtained.

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1 Introduction

The main risk facing investors in Commercial Mortgage-Backed Securities (CMBS) is the risk that borrowers will default on the underlying securitized mortgages. Default erodes a CMBS bond's collateral and depending on the bond structure, can either directly reduce a bond's principal claim, as in the case of the first-loss bond, or reduce the subordination levels of higher rated bonds.

Although there have been numerous academic commercial mortgage default studies using life insurance company data, it is not clear their estimates are applicable to CMBS conduit loans. (See, for example, Vandell (1992), Vandell, Barnes, Hartzell, Kraft and Wendt (1993), Ciochetti, Deng, Gao and Yao (2002) and Archer, Elmer, Harrison and Ling (2002)). More recent studies, such as Ambrose and Sanders (2003) and Chen and Deng (2004) have started using data from CMBS mortgage pools. Ambrose and Sanders (2003) investigated prepayment and default as competing risks using information from over 4000 loans originated during the 1990s while Chen and Deng looked at servicers' abilities to avert losses when a loan defaults. There have also been numerous industry studies investigating conduit loan defaults. (See, for example, Harrison and Story (2004), Hu and Chun (2003), O'Rourke and Metz (2004), Parkus and Tierney (1999) Pelletier and Rudenstein (1998), Rubinstein and Todd (2004) and Stouse, Anderson, Telfeyan and Overby (2004)). Most of these studies have simply tabulated results and none of them (with the possible exception of Stouse, et. al.) attempted to estimate a default model. A potential difficulty, common to all the cited studies, is that default is defined as loans which were at least 30 days, and sometimes 60 days delinquent. Since not all delinquencies result in default, such studies may overestimate the incidence of default. Furthermore, many delinquent loans become current after the initial delinquent period. A difficulty with the academic studies is that default rates are based on different property types which presumably have different default parameters. Furthermore, to the best of our knowledge, no study has investigated the pattern of delinquencies as well as the various stages of loan distress and whether such information is informative in predicting loan default.

In this paper we propose a structural model of delinquency and defaults assuming perfect state information. While this assumption is perfectly unrealistic, it gives us the advantage of building the different components of the model and analyzing how they interact with each other. It also helps in developing a sense for how the optimal policy is shaped. In a following step we propose two different approaches to estimate the model. In place of perfectly observing the different states, we acknowledge that we only receive some observations about the value of the current state, which may be corrupted by stochastic uncertainty. These estimation methods deal with the fact that we have unperfect state information differently. In a first approach we use reduced form models and estimate, in a more conventional manner, a hazard model for the various levels of distress we observe. A second approach focuses on the structural estimation of the discrete decision process by estimating the conditional choice probabilities by the method of maximum likelihood. We capture the exiting dynamics in the borrowers sequence of payment decisions under uncertainty by estimating a model of default that is consistent with a dynamic programming behavioral model of borrower's delinquency and default. Before carrying out the estimation procedures, we present the data and the different variables used in the model. In a final step we carry on these two estimation methods outlined focusing only on securitized hotel mortgage loans. There are several reasons for our hotel loan focus.

Among securitized loans, hotels loans have experienced the highest incidence of defaults. For the period from 1993 to 2003, 303 hotel loans representing 15.9% of all securitized hotel loans defaulted (Harrison and Story, 2004). In contrast, only 3.8% of retail and 2.8% of office loans defaulted over the same period.

By focusing on one property type, this controls for possible variations in parameter estimates as a result of property heterogeneity. Furthermore, by focusing on the highest risk property type, our estimates can be interpreted as upper bound for the default risk factors which may be useful in understanding default for all loan types.

A unique feature of our data is the possibility of following each borrowers payment history and to distinguish between the various resulting stages of loan distress. We have observations on hotel loans which have missed one, two and many more payments resulting in 30-days, 60-days 90-days delinquencies, loans in foreclosure, loans in REO and those which resulted in an actual foreclosure sale. This last category is particularly important since it allows us to determine loss severity information of such sales which are critical in assessing credit default risk of CMBS bonds. Furthermore, we have information on the number of times distressed loans have been delinquent and the pattern of delinquency prior to each subsequent stage of distress. Since the primary focus of our study is a model that determines the factors leading to default, we can determine whether the pattern of delinquency is a predictor of the next level of distress - that is, the conditional probabilities for each distress levels.

Another advantage of focusing on hotel mortgages has been our ability to obtain detailed market data from Smith Travels Research (STR) which provides contemporaneous market data such as vacancy rates and the average daily rates of each hotel at the time of delinquency or default. Its inclusion provides timely market information of comparable hotels in the market areas of the troubled hotels. This data allows us to construct better contemporaneous debt service coverage ratio. Past studies have either used debt coverage ratios at origination or ratios constructed from regional aggregate indexes such as the NACREIF NPI index. We do not believe such detailed contemporaneous micro-level data exists for any other property types.

2 Summary of the Different Resolution Methods

A recent joint research effort by Moody's and Lehman Brothers has documented the different liquidation resolution methods in defaulted CMBS loans. In particular it becomes obvious that borrowers have notoriously been considering different options when dealing with their properties debt payments. These decisions are of prime importance for bondholders since each path taken can lead to different resolution methods and hence have considerable weight on the amounts recovered. They most often manifest through different delinquency paths. Understanding the existing relationship between equity shortfalls, prepayments and delinquency becomes indispensable if one wants to model the credit worthiness of these loans.

The following resolution methods have been observed in the CMBS universe.

Resolution with material loss: For the most part, these are loans that have been liquidated, resolved through discounted pay-offs or a third-party note sale. They are in general characterized by heavy loss severities. They most likely result from an equity shortfall. In some cases however, loans are liquidated without incurring significant, if any losses. Such instances are less common but reflect the repeated borrower's inability to make payments on time. Equity remains however strong enough that other investors seize the asset without a discount, hoping to implement better cash flow policies. In the case of an equity shortfall, we expect to observe a straight shot delinquency path. Once the borrower has contemplated every option and acknowledged that the loss in equity outweighs all other options, she stops making payment and walks towards liquidation. On the other hand, weak market performances combined with low levels of cash in hands will trigger erratic delinquency patterns. There is very little prediction possible in this case as this could be a lengthy process where cash reserves could recover for a while, but never enough to fully cure. Nevertheless, equity need not be affected immediately by this economical decay. In fact, in spite of poor market performance, borrowers probably decide to hold on to their property and struggle to make payments on time only because of a significant equity stake remaining available.

Resolution with minimal loss: This category essentially includes cases where the special servicers' fees have been processed within the trust as realized losses, as opposed to being paid out of interest cash flows. Resolutions with minimal losses remain marginal in their magnitude. They have also nothing to do with the borrowers' behavior except for the fact that they have most likely not been triggered by an equity shortfall. No particular delinquency patterns can be inferred.

Resolution with no losses: These are loans that in spite of being materially delinquent when they paid-off, have taken absolutely no losses. In these cases, loans have been fully prepaid, probably as a consequence of advantageous refinancing options. While most, if not all, securitized loans in the CMBS universe are underwritten with strict prepayment penalties, it is worth noting that, as documented by Moody's, in about 70% of these resolutions none or only part of the prepayment penalty was collected. It has been brought to our attention that delinquency is strategically used by borrowers as a threat. The aim is to bring the lender to let go of the prepayment penalties to avoid a costly liquidation. For the threat to be effective it must be credible. The most obvious threat for a lender could be a decline in property value, which could trigger a poor recovery. Note though that the property value must remain above the threshold set for defaults under optimality conditions, otherwise the borrower would not seek to refinance. Under the conditions set forth above, we expect to observe short term delinquencies resulting rapidly in full liquidations.

3 A Dynamic Programming Representation of Delinquency and Default

3.1 The Optimal Time To Default

In this section we analyze the problem of determining the optimal default policy in a dynamic framework. We consider the case of an equity holder facing the prospect of deciding when, if ever, to default. We assume for simplicity matters that the borrower has risk neutral preferences. In this simple context, the default policy is defined as the argument that maximizes the expected reward from defaulting. In particular, the stopping property value can be perceived as the boundary beyond which the expected reward from defaulting exceeds the expected future reward if it is chosen to not default today. Property value and interest rates are assumed to be stochastic albeit independent from each other.

This approach is closely related to the model first presented by Leland and Toft (1996) and later modified by Duffie and Lando (2001). It differs however at least in two fundamental aspects. In these two papers, the approach taken follows a corporate finance framework in which the enterprise value is set as the sum of the unlevered assets and the tax shield. We, on the other hand, chose to model the property value independently from the free cash flows generated from operations and capital structure decisions. While in theory, there indeed should exist a relationship between the amount recovered from the sale of the property and the expected future net operating income, we claim that in practice this relationship need not exist over a short term horizon. We henceforth model NOI and property value as two independent stochastic processes. For simplicity matters we chose not to include the impact of taxes on enterprise value and leave the capital structure issue for a more comprehensive model.

Suppose $H(t)$ is the price at time t of a property. We define the reward from defaulting by

$$y_t = M_t(r) - H(t) - C$$

where $M_t(r)$ is the mortgage balance at time t with rate r and C is some finite cost to default. If we decide to default at time t we hence get the expected discounted net payoff at time n

$$J^t(x_t) = E_r\left[e^{\int_t^n r_s\,\mathrm{d}s}(M_t(r) - H(t) - C)\right] \tag{3.1}$$

The borrower must decide whether or not to default today or delay the decision until next period. The control at time t, d_t, is thereafter set to

$$d_t = \begin{cases} 0 & \text{if decide to default at time t} \\ 1 & \text{otherwise} \end{cases}$$

The state variable x_t becomes therefore naturally

$$x_{t+1} = \begin{cases} D & \text{if } x_t \ = \ \text{D} \\ D & \text{if } d_t \ = \ 0 \\ y_t & \text{otherwise} \end{cases}$$

where D is the default state. The dynamic programming algorithm is therefore

$$
\begin{aligned}
J_N(x_N) &= \begin{cases} x_n & \text{if } x_n \neq \text{D} \\ 0 & \text{otherwise} \end{cases} \\
J_t(x_t) &= \begin{cases} \max\Big[J_t(x_t), E_y\big[J_{t+1}(x_t)\big]\Big] & \text{if } x_t \neq \text{D} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The optimal default policy is consequently to default whenever

$$M_t(r) - H(t) - C \geq \frac{E_y\big[J_{t+1}(x_t)\big]}{E_r\big[e^{\int_t^n r_s \,\mathrm{d}s}\big]}$$

which is equivalent to

$$H(t) \leq M_t(r) - C - \frac{E_y\big[J_{t+1}(x_t)\big]}{E_r\big[e^{\int_t^n r_s \,\mathrm{d}s}\big]} \tag{3.2}$$

The optimal policy hence suggest that the borrower will default whenever the reward from defaulting is greater than the expected present value of the expected revenue without defaulting today.

3.2 The Optimal Payout Policy

When the option to default is foregone and the borrower decides to keep the property performing for at least one more month, there remains one important decision to be made: How much should be paid back today? Assuming that the borrower is well informed of her financial situation, she must weight in the different payout options against their future consequences. For example, if she decides to not make her debt service payment this month, she will immediately benefit from it by freeing some cash that in turn she will be able to allocate elsewhere over a short period of time. On the other hand, the loan will become delinquent and the missed mortgage payment will still have to be made at a later date, unfortunately for her, on top of the future mortgage payment that will keep pilling up. Of course, when contemplating which amount to pay, the borrower takes into account her present financial situation. Additionally, she also must look ahead and assess how is that situation going to evolve tomorrow. Future outcomes, such as future operating income, property value or interest rates are to a certain extend random. A *good* payment decision, and we will define what we mean by good, will therefore incorporate this uncertainty in order to assess what debt service payment is best for the borrower right now. One can rapidly realize that the problem faced here is really about finding the optimal timing for each asset allocation. If we are able to define a cost structure that will make correspond, every month, to every possible state of her financial situation and to every payment decision she makes a given value, then we can see that her problem is really an intertemporal cost-minimization problem in a stochastic framework. This makes the Dynamic Programming paradigm a very natural candidate to build the structural model of the borrowers optimal payment decisions.

Before defining rigourously the model and outlining its optimal payout policy, we propose a heuristic representation of the current issue. Consider the situation of a reservoir in a water resource system. Assume that it is big enough so that the maximum capacity is not an issue. Every month, an uncertain quantity of precipitation will raise the level of water in the tank. On the other side a known amount of water must be released for irrigation purposes and for human consumption. In this simplified model, humans cannot survive very long without water. Water must therefore always be available for drinking consumption. On the other hand, crops are more resistant and can survive a temporary interruption of the necessary irrigation. Obviously though, if irrigation is foregone for too long then the crops will wilt and die. As a consequence, the question faced every month by the management of this reservoir is to decide how much water should be released to the population and how much, if any, should be released for agricultural purposes.

Consider now the case where after either a long drought or a series of ill decisions, the water level drops significantly. Before the reservoir dries up completely, in a well informed society, fear will spread amongst the population and one is to expect some popular manifestation of this concern. On the other hand if rainfalls are over-abundant then one might suggest that the reservoir itself was unnecessary and its construction a waste of money. The point we are trying to make is that both a very low or very high volume of water in storage can account for additional costs. This automatically raises the question of a specified optimal volume of water in storage.

We can now use the intuition gathered from this simplified model to determine the variables that really affect structurally every decision. The volume of water in storage in the reservoir can be regarded as the wealth of the property. Rainfalls would consequently translate into sales and human consumption into operating expenses. The deterministic nature of the agricultural needs becomes the mortgage schedule where amounts to be paid back are known in advance. Similarly the maximum amount of time the crops can survive without irrigation can be regarded as the allowance period before the asset is liquidated. Finally in this context, one can see how payment decisions will be based on not only the randomness of the future incomes but also on a given wealth and delinquency level.

3.2.1 A Dynamic Programming Representation

In the first part of this section we define the different variables to be considered as well as the cost functions. We consider the problem of paying a quantity at each of n periods so as to meet a deterministic demand with stochastic inputs, while minimizing the incurred expected cost. For this purpose we define by w_t, u_t and ε_t respectively the level of wealth, the payment decision and the net operating income at time t. We assume that $\varepsilon_0,\ \varepsilon_1, ...,\ \varepsilon_{n-1}$ are independent random variables. Wealth is therefore defined iteratively over time by:

$$w_{t+1} = w_t - u_t + \varepsilon_t \tag{3.3}$$

We define next by e_t and μ_t the equity level and the increment in property value as a result of natural fluctuations in real estate prices at time t. We assume as previously that the increments μ_t are IID. With the notation introduced above, we observe that e_t is defined as

$$e_t = H_t - M_t \tag{3.4}$$

where H_t was defined as the property value and M_t as the remaining mortgage balance at time t. Naturally a payment decision u_t will decrease the mortgage balance by the principal component of the debt service at time t and thus increase the equity stake accordingly. If we define by PV_t^* the function that maps a payment made, given some underwriting criteria *, into its principal component, we can describe the equity level iteratively as follow

$$\begin{aligned} e_{t+1} &= H_t - M_t + PV_t^*(u_t) + \mu_t \\ &= e_t + PV_t^*(u_t) + \mu_t \end{aligned}$$

As a final comment, note that while we assume independence within the ε_i's and within the μ_i's, we make no such assumption between these two stochastic processes. At any time t we allow for some dependance between ε_t and μ_t.

At this point we make an important change of variables and normalize all quantities by the amount of the debt service. While this change of numeraire might seem arbitrary, it greatly reduces the dimension of the state space. For instance a wealth level of 12 truly implies that $w_t = \big(12 \times \text{Debt Service}\big)$ in dollars. Similarly under this change of variables, $\varepsilon_t = DSCR$ and u_t is an integer set to n when n monthly payments are made. For example by $u_t = 2$ we understand that the borrower will be making 2 monthly payments during period t.

We define next the delinquency level by d_t. This delinquency level has a natural iterative definition. It is a summary statistic that compares for every time t the amount of payments made with the amount of payment scheduled. It is defined by

$$d_t = t - \sum_{k=0}^{t} u_k \tag{3.5}$$

which is equivalent to

$$d_t = d_{t-1} + 1 - u_t \tag{3.6}$$

We illustrate how the delinquency level captures the relevant information with a simple example. Assume that at time t a borrower has been making all her payments on time except in two instances where she did not pay. By (3.5) we determine that $d_t = t - (t-2) = 2$ since we had $u_s = 1$ for all $s \leq t$ except for some s_1, s_2 in $\{0, 1, ..., t\}$ where $u_{s_1} = u_{s_2} = 0$. Note that the map d is not bijective since two different delinquency paths can result in a same d_t. While some characteristics of the information set are therefore lost, we claim that d_t as defined is sufficient to characterize the optimal payout policy.

Before defining the cost functions we set the state space to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$ where $\mathbb{N}$ is defined as the set of non negative integers. A state i is thus defined by $i = \big(w, e, d, t\big)$. We can reduce further the state space and specialize it along a given wealth, equity and delinquency trajectory by setting $i_t = \big(w_t, e_t, d_{t-1} \ / \ t \ \in \ [0, n]\big)$. The randomness in ε_t and μ_t induce an uncertainty as of the destination state j_{t+1} when the borrower is in state i_t and exercise a control u_t. We consequently define the probability of a transition from state i_t to state j_{t+1} given u_t by $p_{ij}(u_t) = p\big((w_{t+1}, e_{t+1}, d_t)/(w_t, e_t, d_{t-1}, u_t)\big)$

The cost function considered weights in the particularity of any given state i in three separable ways. On one hand a cost is incurred only when the loan becomes delinquent. This cost will reflect either the loss of control over the borrowers equity as the loan is increasingly delinquent or the benefit gained from freeing some wealth as a result of missed payments. In a latter part of this paper we will consider the case where the delinquency level d_t exceeds some boundary and triggers thereby the liquidation of the property. When such a case is reached, the remaining equity available to the borrower will have been reduced by, at least, the remaining mortgage value. This decrease in equity value is thus being incurred smoothly over the delinquency path in the following way

$$c_{ij}^{d}(u) = \begin{cases} C^{del} + \theta_{12} d_t & \text{, if } d_t > 0 \\ 0 & \text{, otherwise.} \end{cases} \tag{3.7}$$

where C^{del} is the unobervable cost of being delinquent and θ_{12} is known. This cost is incurred, when starting from a state i and making a payment u, state j is reached. On the other hand, as in the reservoir schematic example, we admit that to any level of wealth w_t corresponds a cost $c^{w}(i)$. This second component of the cost function attempts to control for both opportunity and distress costs. The implicit assumptions are that a large level of wealth available can be perceived as either deep pockets from which payments can be drawn in economical downturns or an opportunity cost suggesting this wealth is not being optimally used. Similarly, low levels of cash available can boost distress risk and raise costs accordingly. During the estimation of the model, we investigate several functional forms. A few examples of cost functions mapping an initial wealth level w_t, a payment u_t and an operating income ε_t into its corresponding cost are:

$$1.\ \ c_{ij}^{w}(u) = \theta_{21} + \theta_{22}\big(w_t - u_t + \varepsilon_t\big) + \frac{\theta_{23}}{w_t - u_t + \varepsilon_t} \tag{3.8}$$

or,

$$2.\ \ c_{ij}^{w}(u) = \theta_{21} + \theta_{22}\big(w_t - u_t + \varepsilon_t\big) + \theta_{23}\sqrt{w_t - u_t + \varepsilon_t} \tag{3.9}$$

where the parameters $\theta_2 i$ are known to the borrower. Lastly, we are also concerned about assessing the incidence a given equity level can have on the sequential decision process. For that matter we will, as with the wealth level, investigate several functional forms. Examples include, but are not limited to

$$1.\ \ c_{ij}^{e}(u) = \theta_{31} + \theta_{32}\big(e_t + PV_t^{*}(u_t) + \mu_t\big) + \frac{\theta_{33}}{e_t + PV_t^{*}(u_t) + \mu_t} \tag{3.10}$$

or,

$$2.\ \ c_{ij}^{e}(u) = \theta_{31} + \theta_{32}\big(e_t + PV_t^{*}(u_t) + \mu_t\big) + \theta_{33}\sqrt{e_t + PV_t^{*}(u_t) + \mu_t} \tag{3.11}$$

To completely characterize the cost function we must finally consider the case of default described in the previous section. In case of default, the borrower will, as in the case of delinquency, incur a cost that accounts for a downgrade in credit quality and hence higher future cost of borrowing. In our framework, a decision to default can occur at any time. It is however binding and implies that all future payments will be foregone. During a grace period τ the servicer lets the borrower collect all operating income without making any debt service payments. It is only once the delinquency level has exceeded the grace period that the property is foreclosed. In most cases, a default results also in a loss of equity. By letting $E_t^\tau[w_t]$ and $E_t^\tau[e_t]$ be respectively the expected increments in wealth and property value between time t and $t+\tau-d$ with respect to the joint distribution of ε and μ we define the cost of default as

$$c(default) = \theta_{11} + \theta_{12}\tau + c_{ij}^w\big((d-\tau)E_t^\tau[w_t]\big) + \theta_{32}(\tau-d)E_t^\tau[e_t] + C^{def} \tag{3.12}$$

where C^{def} is the unobservable cost of default and θ_{33}, θ_{31} have been set to zero in order to simplify the notation. We can now combine (3.7), (3.8), (3.10) and (3.12) to define the cost function g as follow

$$g_{ij}(u) = \begin{cases} c_{ij}^d(u) + c_{ij}^w(u) + c_{ij}^e(u) & \text{, if } u \in \{0,1,2,\dots\} \\ c(default) & \text{, if } u = \text{default} \end{cases} \tag{3.13}$$

It will also be useful to define the expected cost functions. With (3.13) we can define

$$E_{\mu,\varepsilon}[g_{ij}(u)/i,u] \quad = \quad g(i,u) = \int_j g_{ij}(u)d[p_{ij}(u_t)] \tag{3.14}$$

In this context the borrower will chose a sequence of feasible decisions $\{u_t\}$ that minimizes the expected cost, i.e.

$$\{u_t^*\} = \underset{u_t\,\epsilon\,U}{\operatorname{argmin}}\, \mathrm{E}_{\mu,\varepsilon}\Big[\sum_{t=0}^{n-1}\big(g_{ij}(u_t)\big)\Big] \tag{3.15}$$

or equivalently with the notation introduced in (3.14) we can pose our problem in terms of determining the optimal payment quantity $\{u_t^*\}$ such that the expected value of all cost incurred over the horizon is minimal. The objective function is as follow

$$J_o(i_o) = \sum_{t=0}^{n-1}\big[g(i_t, u_t(i))\big] \tag{3.16}$$

Invoking Bellman's principle of optimality, we can introduce the following functional equations of dynamic programming associated with problem (3.16)

$$\begin{aligned} J_n(i_n) &= 0 \\ J_t(i_t) &= \min_{u_t\,\epsilon\,U}\Big[g(i_t,u_t(i)) + \int_j J_{t+1}(j)d[p_{ij}(u_t)]\Big] \\ U &= \big\{u_t \mid 0 \le u_t \le w_t \text{ and } 0 \le d_t \le \tau\big\} \end{aligned} \tag{3.17}$$

4 Estimation of the model

In this section we propose a method for estimating the structural model for default that was previously analyzed. Broadly speaking we have two main options to do this. The inherent complexity of the model pushes the statistician to make several approximations to make the estimation feasible. A popular approach is to work with closed form formulas in order to estimate a vector of parameters that best explains data pairs by maximum likelihood. In this case one very conventional method is to model intensities in a reduced form model such as a hazard model. Another option available is to derive a model that is consistent with the dynamic programming paradigm detailed previously. In particular John Rust derived a nested fixed point algorithm that takes advantage of the contraction mapping properties of the DP algorithm. Under some conditions to be detailed further, it can be shown that the conditional probability of observing some controls given a particular state is the well known multinomial logit formula. It is therefore possible to estimate the maximum likelihood parameters by nesting together a gradient type of algorithm with a poly-algorithm that iterates the value of the cost-to-go function for any given parameter value. We present next the maximum likelihood method, namely the Nested Fixed Point Algorithm.

4.1 Maximum Likelihood Estimation

In this section, we show how the methodology established by John Rust can be used to estimate our model. We give here a brief summary of the procedure but refer the reader to the corresponding papers for further details.

The main difficulty in econometrics methods for discrete decision processes is the lack of first order necessary conditions to characterize the stationary optimal decision rule $\delta^* = \{\delta, \delta, ..., \delta\}$. We are instead restricted to a system of inequalities where the sub-optimal allocation is characterized by:

$$u \,\epsilon\, \delta(i) \Leftrightarrow \Big\{\forall\ u' \epsilon\ U(i) \mid u(i) + g(i, u(i)) + \int_j J(j) d[p_{ij}(u)] \geq u'(i) + g(i, u'(i)) + \int_j J(j) d[p_{ij}(u')]\Big\}$$

A second difficulty arises as a consequence of not observing completely the states at which the system is. This is in fact one of the two major caveats of structural models. One popular method is to use reduced form models. Another is to assume that these unobserved variables of the state space follow a given distribution. For instance, consider the problem of the optimal payment policy analyzed in section 3.2 on page 5. In particular if we observe panel data $[w_t^a, e_t^a, d_t^a, u_t^a]$, indexed by a if we have several individuals, then we can set $i = [w_t, e_t, d_{t-1}, \nu_t]$ where $[w, e, d]$ are observed by both the econometrician and the agent and ν is known by the agent only. To simplify notation we let $x_t = [w_t, , e_t, d_{t-1}]$. In such framework the decision rule $\delta(x_t, \nu_t)$ becomes deterministic and we can define the conditional choice probability as:

$$P(u/x) = \int \mathrm{I}_{\{u\,=\,\delta(i)\}} q(d\nu/x) \tag{4.1}$$

where $q(d\nu/x)$ is the conditional distribution of ν given $[w, e, d]$. One important aspect is to specify necessary assumptions such that $\delta(x_t, \nu_t)$ is not degenerate. A specification for unobservables is saturated if the following holds

$$u \ \epsilon \ \delta(i) \Leftrightarrow P(u/x) > 0$$

The three following assumptions guarantee that the conditional choice distribution has full support.

1. *The state variables obey a markov process:* the transition density is given by a parametric function $\pi(dx_{t+1}, d\nu_{t+1}/x_t, \nu_t, u_t, \theta)$ where θ is a vector of unknown parameters.

2. *The cost function is additively separable:* $g_{ij}(u) = g(x_t, u_t, x_{t+1}) + \nu_t(u_t)$ where $\nu_t(u_t)$ is the u^{th} component of the vector ν.

3. *The transition matrix is conditionally independent:* the transition matrix π becomes $\pi(dx_{t+1}, d\nu_{t+1}/x_t, \nu_t, u_t, \theta) = p(dx_{t+1}/x_t, u_t, \theta)q(d\nu_t/x_t, \theta)$

Assumption [1] and [2] together with equation (3.17) give us a new functional system of equations associated with problem (3.16).

$$\begin{aligned} J_t(i_t, \theta) \quad &= \min_{u_t \ \epsilon \ U(x_t)} \quad \Big[g(x_t, u_t) + \nu_t(u_t) \\ &\qquad + \beta \int_{(x_{t+1}, \nu_t)} J_{t+1}(x_{t+1}, \nu_t, \theta) \pi(dx_{t+1}, d\nu_{t+1}/x_t, \nu_t, u_t, \theta) \Big] \end{aligned} \quad (4.2)$$

If we denote EJ_θ by

$$EJ_\theta(x, \nu, u) = \int_{(x', \nu')} J_\theta(x', \nu', \theta) \pi(dx', d\nu'/x, \nu, u, \theta)$$

where we have dropped the time indices for notational simplicity, then we can rewrite (4.2) in the compact form:

$$J_\theta(i) = \min_{u \ \epsilon \ U(x)} \Big[g(x, u) + \nu(u) + \beta EJ_\theta(x, \nu, u) \Big] \quad (4.3)$$

We are now ready to specify the maximum likelihood function. Let $L(x_1, ..., x_T, u_1, ..., u_T/\theta)$ be the joint probability distribution of the observed data $[x_t, u_t]$. The maximum likelihood estimator is denoted as θ^* where $L(\theta^*) = L(x_1, ..., x_T, u_1, ..., u_T/\theta^*)$ is maximum. Under the further assumption that $q(d\nu/x)$ is an IID multivariate extreme value distribution, then the likelihood function can be shown to be equivalent to [1]

$$L(x_1, ..., x_T, u_1, ..., u_T/\theta) = \prod_{t=2}^{T} P(u_t/x_t, \theta) p(dx_t/x_{t-1}, u_{t-1}, \theta) \quad (4.4)$$

where $P(u_t/x_t, \theta)$ is given by the multinomial logit formula

$$P(u/x, \theta) = \frac{\exp\big[g(x, u, \theta) + \beta EJ_\theta(x, u)\big]}{\sum_{u' \varepsilon U(x)} \exp\big[g(x, u') + \beta EJ_\theta(x, u')\big]} \quad (4.5)$$

[1] see McFadden, 1973

and where the cost-to-go EJ_θ is the unique fixed point to the contraction mapping defined by $T_\theta(EJ_\theta) = EJ_\theta$, that is:

$$\begin{aligned} EJ_\theta(x,u) &= T_\theta(EJ_\theta)(x,u) \\ &= \int log \left[\sum_{u' \varepsilon U(x')} exp\big[g(x',u') + \beta EJ_\theta(x',u')\big] \right] p(dx'/x,u,\theta) \end{aligned} \tag{4.6}$$

John Rust showed that T_θ is indeed a contraction mapping. Moreover consistent estimates of θ can be obtained by maximizing the partial likelihood function

$$L_p(\theta) = \prod_{t=1}^{T} P(u_t/x_t, \theta) \tag{4.7}$$

Finally, notice that estimation by maximum likelihood is equivalent to finding a value of θ that maximizes either $L(\theta)$ as defined in (4.4) or $L_p(\theta)$ given in (4.7) provided that the function EJ_θ is the unique fixed point to the system of equations (4.6).

4.2 The Nested Fixed Point Algorithm

The simplest method to estimate θ^* is to take advantage of the convergence properties of the contraction mapping by value iteration. Existence and uniqueness of EJ_θ fully characterize the function implicitly. For any θ it is therefore possible to derive the corresponding value of EJ_θ. A nested fixed point algorithm will consequently nest two complementary algorithms. One *outer* optimization algorithm that maximize $L_p(\theta)$ with respect to θ and one *inner* fixed point algorithm that computes EJ_θ for some θ.

4.2.1 *Inner* fixed point algorithm:

Traditionally fixed point estimation has been carried out by value iteration algorithms where the contraction mapping T_θ is run iteratively over some initial guess of the cost-to-go EJ_θ^0. This methods converges in the limit to the true value of EJ_θ^* for *any* initial conditions. Value iterations' main drawback is that convergence can be very slow when the modulus of the *subdominant* [2] eigenvalue of the transition matrix P_θ, whose ij^{th} entry is $p(dx_j/x_i, u_i, \theta)$, is close to 1. An alternative is embodied in policy iteration algorithms which converge very quickly to EJ_θ^* for any *educated* initial guess. The nested fixed point algorithm combines these two methods, where value iteration is carried out initially until EJ_θ^k is within some neighborhood of the true value. Only then the procedure switches to policy iteration to converge, in very few steps, to its limit point.

Note that the observed variable $x_t = \{w_t, e_t, d_t\}$ was up to now assumed to be continuous. In particular there were no restrictions imposed, other than being non negative, to the wealth level w or the equity level e. This alone implies that the required fixed point is infinitely dimensional, which makes its numerical derivation impossible. We must henceforth discretize the

[2] The eigenvalue with second largest modulus

state space so that the wealth and equity level takes only finitely many values. In essence, this simplification amounts to approximating the problem of deriving the fixed point to a contraction mapping T_θ in an infinite dimensional space by the problem of deriving the fixed point to another contraction mapping $T_{n,\theta}$ on a high dimensional Euclidean space.

One such way to discretize for example the wealth level is to set the new variable $\widetilde{w}$ as

$$\widetilde{w} = \begin{cases} 0 & \text{, if } w \ \varepsilon \ [-\infty, \frac{1}{2}) \\ n & \text{, if } w \ \varepsilon \ [n - \frac{1}{2}, n + \frac{1}{2}) \\ N & \text{, if } w \geq N \end{cases}$$

With this newly defined wealth level, we obtain the corresponding observed variable $x = \{\widetilde{w}, \widetilde{e}, d, u\}$. Accordingly the transition probability function $p(x'/x, u, \theta)$ becomes an $N \times N \times N$ markov transition matrix and the contraction mapping T_θ defined in (4.6) becomes

$$\begin{aligned} EJ_\theta(x,u) &= T_\theta(EJ_\theta)(x,u) \\ &= \sum p(x'/x,u,\theta) \ log \left[\sum_{u' \varepsilon U(x')} exp\big[g(x',u') + \beta EJ_\theta(x',u')\big] \right] \end{aligned} \tag{4.8}$$

With the previous structure we give a formal description of the value iteration and the policy iteration algorithms.

1. *Value iteration (successive approximation):* For any θ the algorithms updates the k^{th} estimate by the following procedure,
   - Start with an arbitrary initial guess, say $EJ^0 = 0$.
   - Generate every update by successive approximation $EJ^{k+1} = T_\theta(EJ^k)$. Note that we can write the k^{th} stage by iteratively substituting as a function of the initial guess, i.e. $EJ^{k+1} = T_\theta^k(EJ^0)$.
   - As $k \to \infty$ we have $EJ^k \to EJ_\theta$ in the sense of the sup norm.
2. *Policy iteration (Newton-Kantorovich iteration):* The idea is to convert the original problem of finding a fixed point to (4.8) into finding a solution to the equation $[I - T_\theta](EJ_\theta) = 0$ where I is the identity operator.
   - It can be shown that the non-linear operator $[I - T_\theta]$ accepts a Frechet derivative $[I - T_\theta^{'}]$.
   - Since $[I - T_\theta^{'}]$ is invertible, we can do a Taylor expansion about the k^{th} fixed point estimate of the equation $[I - T_\theta](EJ_\theta) = 0$.

     $$[I - T_\theta](EJ^k) + [I - T_\theta^{'}](EJ^{k+1} - EJ^k) \sim [I - T_\theta](EJ_\theta) = 0$$

   - We finally solve for EJ^{k+1} since $[I - T_\theta^{'}]$ is invertible and obtain the Newton-Kantorovich iteration:

     $$EJ^{k+1} = EJ^k - [I - T_\theta^{'}]^{-1}(I - T_\theta)(EJ^K)$$

In conclusion, we proceed as mentioned previously. We begin by deriving successive estimates of the expected cost-to-go by value iteration. Since the convergence is only linear and henceforth slows down significantly as you approach EJ_θ we switch to the Newton-Kantorovich iteration as soon as the expected cost-to-go is within some distance of EJ_θ in order to converge quadratically to its true value.

4.2.2 *Outer* optimization algorithm:

The optimal θ^* is derived with a standard nonlinear maximum likelihood optimization algorithm known as BHHH accelerated by the BFGS (Broyden, Fletcher, Goldfarb and Shannon) method. Note that the BHHH algorithm was named after the article by Berndt, Hall, Hall and Haussman, 1976, *Estimation and Inference in Nonlinear Structural Models.* It is a quasi-Newton gradient optimization algorithm that chooses a search direction using the outer product of the gradient of the log-likelihood function as an approximation to the negative of the hessian matrix. The parameter iteration is

$$\theta^{k+1} = \theta^k + \lambda_k D(\theta^k)$$

where λ_k is a step size that is chosen so as to maximize the steepest descent in the likelihood function and $D(\theta^k)$ is a descent direction. Note that if, for all k

$$\begin{aligned} \lambda_k &= 1 \\ D(\theta) &= -\frac{\partial L(\theta)/\partial\theta}{\partial^2 L(\theta)/\partial\theta\partial\theta'} \end{aligned}$$

then BHHH reduces to Newton's method. Each iteration should therefore require the estimation of $\partial^2 L(\theta)/\partial\theta\partial\theta'$, which is clearly non-trivial. Instead of deriving the negative of the hessian, the BHHH methods approximates $D(\theta)$ by the Fisher information matrix $I(\theta)$. In particular it is well known that under smoothness conditions on $L(\theta)$ we have the following equality

$$I(\theta^*) = -E\big[\partial^2 L(\theta^*)/\partial\theta\partial\theta'\big] = E\Big[[\partial L(\theta^*)/\partial\theta][\partial L(\theta^*)/\partial\theta']\Big] \tag{4.9}$$

Now if we let $L_m(\theta)$ be the log-likelihood function for a given loan m then, from (4.4), we write $\log L_m(\theta)$ as

$$L_m(\theta) = \sum_{t=1}^{T} \log\left[P(u_t^m/x_t^m, \theta)\right] + \sum_{t=1}^{T} \log\left[p(x_t^m/x_{t-1}^m, u_{t-1}^m, \theta)\right] \tag{4.10}$$

For the whole sample the full log-likelihood becomes then trivially

$$\log L(\theta) = \sum_{m=1}^{M} L_m(\theta)$$

The approximation made by the BHHH method is therefore to take the sample average of the square of the first derivative of the log-likelihood as the direction vector $D(\theta)$. Note that this is reasonable for a large sample and for a value of θ close to the true value θ^*. In place of $-[\partial^2 L(\theta)/\partial\theta\partial\theta']^{-1}[\partial L(\theta)/\partial\theta]$, $D(\theta)$ becomes

$$D(\theta) = \frac{1}{M}\sum_{l=1}^{M}[\partial L(\theta)/\partial\theta][\partial L(\theta)/\partial\theta']$$

where the first order derivatives of the log likelihood function can be calculated analytically using the implicit function theorem. From $EJ_\theta = T(\theta, EJ_\theta)$ we obtain successively

$$\begin{aligned}
\frac{\partial EJ_\theta}{\partial\theta} &= \frac{\partial T_\theta}{\partial\theta} + \frac{\partial T(EJ_\theta)}{\partial EJ_\theta}\,\frac{\partial EJ_\theta}{\partial\theta} \\
\frac{\partial T_\theta}{\partial\theta} &= \left(I - T_\theta^{'}(EJ_\theta)\right)\frac{\partial EJ_\theta}{\partial\theta} \\
\frac{\partial EJ_\theta}{\partial\theta} &= \left(I - T_\theta^{'}(EJ_\theta)\right)^{-1}\frac{\partial T_\theta}{\partial\theta}
\end{aligned} \tag{4.11}$$

Finally, as mentioned previously, the BHHH optimization is accelerated by the BFGS methodology to choose the step size parameter λ that maximizes the univariate function $f(\lambda) = L\big(\theta + \lambda D(\theta)\big)$. The necessary condition is that $\partial f(\lambda^*)/\partial\lambda = 0$. If we try to derive iteratively λ^* by the Newton's method, we get the updating iteration

$$\lambda k + 1 = \lambda_k - [\partial^2 f(\lambda_k)/\partial\lambda^2]^{-1}[\partial f(\lambda_k)/\partial\lambda] \tag{4.12}$$

By a similar argument, we avoid the numerical derivation of the second derivative of $f(\lambda)$, which would require computing the hessian of the likelihood function. In lieu, we use the *secant iteration.* This is simply Newton's method with a finite difference approximation to $\partial^2 f(\lambda_k)/\partial\lambda^2$. The iteration (4.12) becomes thereby

$$\lambda_{k+1} = \lambda_k - \frac{(\lambda_k - \lambda_{k-1})\partial f(\lambda_k)/\partial\lambda}{\partial f(\lambda_k)/\partial\lambda - \partial f(\lambda_{k-1})/\partial\lambda}$$

4.3 Translating the fixed point algorithm into matrix algebra

In order to carry out the estimation of the parameters value, we must translate the nested fixed point algorithm into its matrix analog. We start by showing how the dynamic programming problem of finding the optimal payout policy, laid out in section 3, can be defined in Matlab code.

Maximum likelihood estimation of the optimal payout policy: As we have seen in section (3), the optimal payout policy problem is about deciding how much to pay at every period provided that it has been decided to not default. Our data consists of monthly observations of all hotel loans securitized in the CMBS universe. The data for each loan consists of a vector $\{x_1, \ldots, x_T, u_1, \ldots, u_T\}$ where u_t is set to an integer n when n monthly payments are made at time t or set to default if we the borrower has decided to default instead and where $x_t = \{w_t, e_t, d_{t-1}\}$. The choice set is $U(x) = \{u : u \le \min\{w\ ,\ d+1\}\}$. The cost function was defined in (3.14) and in (4.2) by $\big(g(x,u)+\nu(u)\big)$. The stochastic process governing the evolution of x given by the probability $p(x_{t+1}/x_t, u_t, \theta)$ is

$$p(x_{t+1}/x_t, u_t, \theta) = f\big((w_{t+1} - w_t + u_t \ ,\ z_{t-1} + 1 - u_t \ ,\ e_{t+1} - e_t - PV_t^*(u_t))/\theta\big) \qquad (4.13)$$

In particular notice how x_{t+1} is a combination of a deterministic and two stochastic processes. Whereas $(w_{t+1} - w_t + u_t)$ and $(e_{t+1} - e_t - PV_t^*(u_t))$ are uncertain and closely related to the unconditional distribution of ε_t and μ_t, $d_t = (d_{t-1} + 1 - u_t)$ is completely deterministic and well defined for any d_{t-1} and u_t. In this context the total wealth and equity levels follow a random walk, where the monthly incremental income after debt service payment, namely $w_{t+1} - w_t + u_t$ and where the monthly increment in equity after debt service, $e_{t+1} - e_t - PV_t^*(u_t)$, are distributed according to the density $f(dw + u, de + PV(u)/\theta)$ with support on the interval $[0, \infty)$. This together with the discretization of the state variables $\widetilde{w}$ and $\widetilde{e}$ proposed in (4.8), identifies the transition probability function $p(x_{t+1}/x_t, u_t, \theta)$ as an $N \times N \times N$ Markov transition matrix. In this particular case, the monthly joint wealth and equity distribution $p(x'/x, \mu(x), \theta)$ equals the density $f(w' - w + u, e' - e - PV(u)/\theta)$, which reduces further to a simple multinomial distribution on the discretized state space. For example in the case where $N = 5$ and where ε and μ are distributed according to a 4-nomial distribution on the set $\{0, 1, 2, 3\}$ corresponding to monthly incomes in the intervals $[0, 1)$, $[1, 2)$, $[2, 3)$ and $[3, \infty)$ and equity increments in the same intervals, we obtain a transition probability $\mathrm{Q}_i(w_t, e_t, u_t) = Pr\big(w_{t+1} = w_t + i - u_t, e_{t+1} = e_t + i + PV_t^*(u_t)\big)$, for $i = \{-1, 0, 1, 2, 3\}$. We can represent the discrete transition probability by a series of $N \times N$ Markov transition matrices Π^u given by, for example in this case for $u = 0$ and $u = 2$:

$$\Pi^0 = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & 0 & 0 \\ 0 & \pi_0 & \pi_1 & \pi_2 & \pi_3 & 0 \\ 0 & 0 & \pi_0 & \pi_1 & \pi_2 & \pi_3 \\ 0 & 0 & 0 & \pi_0 & \pi_1 & \pi_2 + \pi_3 \\ 0 & 0 & 0 & 0 & \pi_0 & 1 - \pi_0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Pi^2 = \begin{bmatrix} 1 - \pi_3 & \pi_3 & 0 & 0 & 0 & 0 \\ \pi_0 + \pi_1 & \pi_2 & \pi_3 & 0 & 0 & 0 \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 & 0 & 0 \\ 0 & \pi_0 & \pi_1 & \pi_2 & \pi_3 & 0 \\ 0 & 0 & \pi_0 & \pi_1 & \pi_2 & \pi_3 \\ 0 & 0 & 0 & \pi_0 & \pi_1 & \pi_2 + \pi_3 \end{bmatrix}$$

where for example $\mathrm{Q}_1(1, 2) = \pi_0 + \pi_1$

We are now in position to show how the general equations (4.8) and (4.5) translate into the specific equations of the optimal payout policy. With the newly introduced transition matrices Π^u, it is straightforward to verify that, for an n-nomial distribution of ε and μ, (4.8) can be written as:

$$EJ_\theta\big((w, e, d), u\big) = \sum_{i=0}^{n} \mathrm{Q}_i(w, e, u)\ log \sum_{u' \in U(x')} exp\Big[g\big((w - u + i\ , e + PV(u) + i\ ,\ d + 1 - u), u', \theta\big)$$
$$+ \beta EJ_\theta((w - u + i\ , e + PV(u) + i\ ,\ d + 1 - u), u')\Big] \qquad (4.14)$$

where the condition $u' \in U(x')$ is equivalent to $u' \leq \min\{w - u + i\ ,\ z + 2 - u\}$. This last condition only verifies that the borrower is not paying either more than she owes or more than she has.

5 Data

From 1986 till November of 2004, over \$53 billion of hotel mortgages were securitized. While this is small relative to the total CMBS issuance of \$804 billion over the same period, this represents a sizable proportion of total hotel mortgage originations in recent years. The data consists of a loan level panel data from which we have been able to estimate for each loan the relevant state variables $\{w_t, e_t, d_{t-1}\}$ and their corresponding sequence of controls $\{u_t\}$. There are 1506 loans in total, of which 111 have defaulted.

5.1 Distress Levels and Payment Decisions

We were able to obtain "distressed" and performing loan information from the Bank of America. We had 680 loans that had either been delinquent for 30, 60, or 90+ days, foreclosed, in REO status or sold or disposed of. All loans in our sample were originated from January 1996 till November, 2004. Over this period, there was a total of \$32,327,000 of hotel loans originated and our distressed loan sample accounts for \$4,066,000 or approximate 12.6% of all originations. In our distressed loan sample, 552 loans were 30-days delinquent, 452 were 60-days delinquent, 407 were 90+ days delinquent, 129 were foreclosed, 158 were REO and 220 were sold. Note that these values do not sum to 680 since some loans have experience multiple levels of distress.

From this data we were able to extrapolate a sequence of payment decisions $\{u_t\}$ the following way. For every loan, we decided to infer the logical payment decisions that would result in the several distress levels observed. For example, if a performing loan remained performing over the next month, it was assumed that the borrower made 1 monthly payment over that period. If the status of the loan moved to 30 days delinquent, then it was assumed that she missed that scheduled payment. Similarly, a 30 days delinquent loan that remained 30 days delinquent over the next month resulted in one single monthly payment made. If that same 30 days delinquent loan moved to 60 days delinquent, then we would set u_t to 0, whereas if the loan was cured we would set the payment a time t to 2 monthly payments. For loans that were over 90 days delinquent we have decided to set u_t to 0 unless we were able to observe a downgrade in the distress level. For instance a 90+ days delinquent loan at month t that remained 90+ days delinquent at month $t+1$ would be the consequence of a missed payment. On the other hand, a foreclosed loan with a 8 months delinquency that shifted to a 90+ days delinquency level would result in an hypothesized payment of 6 monthly payments. Our purpose in constructing the sequences $\{u_t\}$ for each loan is to be able to measure the relative importance of the distress level on the sequential decision process. We are however aware that this extrapolation is not perfect and in particular that we might not be tracking accurately the historical payments made for severe distress levels. As a consequence we have decided to run our model with a decision set $U = \{0, 1, 2, 3\}$ corresponding to successively $\{0\}$ no debt service payment, $\{1\}$ one monthly debt service payment, $\{2\}$ two monthly debt service payments and $\{3\}$ three monthly debt service payments. In turn the delinquency set is capped with a maximum of 120 days of late due payments. This way we can assess more precisely the relative weight of the delinquency states in subsequent delinquencies or default. Ideally though, with more precise data, this model can be extended to much higher delinquency levels and payment decisions.

We augment the decision set U by adding to the possible payment options contemplated by the borrower the option to default. As a consequence we construct categorical time series for each loan that we set to 1 when it is assumed that the borrower decided to default and 0 otherwise. For the purpose of our model, we understand default as the decision made by the borrower to forego all future debt service payment regardless of the magnitude of future increments in wealth and property value. We define thus consequently the time of default as the next to last month we were able to observe a payment for a loan that has been liquidated. We admit that with this definition we might, on one hand be missing default decisions from some loans that have indeed decided to default but that have not been liquidated yet, and on the other hand be setting some default earlier that they did happen. With these sequences $\{u_t\}$, we derive the resulting delinquency levels for each loan following (3.5).

5.2 Cumulated Wealth

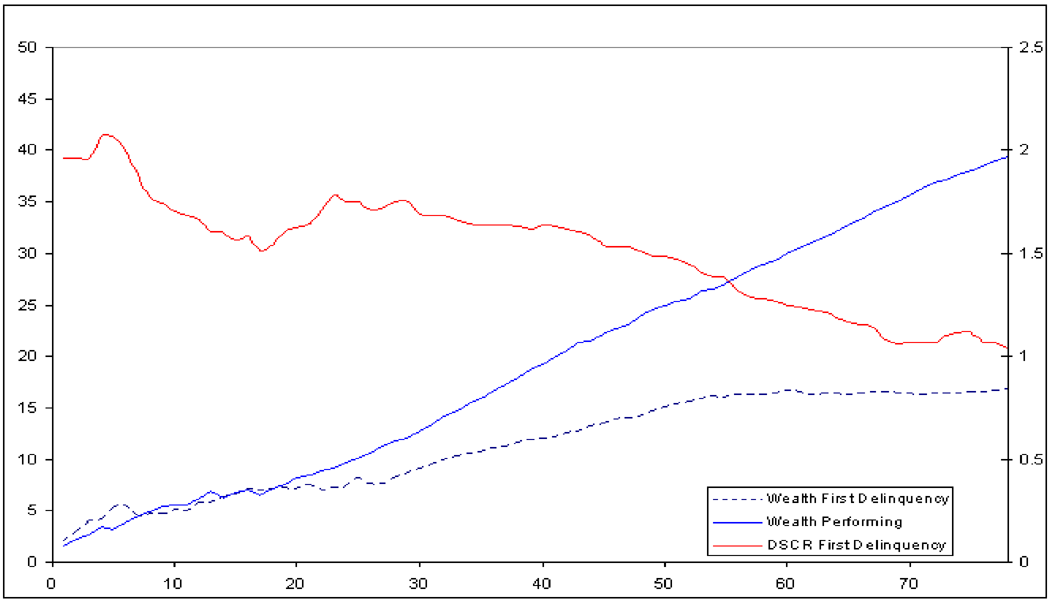
As we have seen previously, the estimation of our model in contingent upon having a reliable estimate of the accumulated wealth w_t of a property at any time t. We construct the wealth variable by in turn matching the Bank of America data set with information from Smith Travel Research (STR). For each loan, the STR data provides information on the individual hotel's performance relative to the performance of similar hotels in the same geographical area. In particular STR compiles occupancy rates and average daily rates for the majority of hotels in the US. We were able to match about 1506 hotels with the STR database and were able to obtain the revenue per available rooms (REVPAR) for the competitive set of hotels for each of our subject properties. This measure takes into consideration both the rates charged as well as the occupancy rate thus is a proxy measure of the performance of hotels in the area. This in turn allows us to determine, at the property level, the performance of any hotel relative to comparable regional properties and to derive the Net Operating Income (NOI) for all the loans included.

From these contemporaneous NOI, following the structural analysis outlined above, we construct the wealth variable. This variable aggregates over time the net operating income minus the debt service payment made. Note that all these amounts are scaled by the scheduled debt service, assuming it remains constant. For instance for a performing loan, at any period t, the contribution to wealth is equal to (DSCR -1). If a loan misses a payment then the contribution to wealth becomes (DSCR - 0). If for some reason a borrower chooses to make two monthly payments at any given point in time, then that month, its contribution to wealth will be (DSCR - 2). We summarize this by formally defining the variable wealth in period t by w_t as

$$\begin{aligned} w_t &= \sum_{i=1}^{t} \frac{\text{NOI}_t - u_t}{\text{Debt Service Scheduled}} \\ &= \sum_{i=1}^{t} \big\{ \text{DSCR}_t - \frac{u_t}{\text{Debt Service Scheduled}} \big\} \end{aligned}$$

where NOI_t is the net operating income in period t and u_t is the debt service payment made in period t. We calculate this for all loans and all observed periods. Note that for a "healthy" property, we expect on average coverage ratios to be greater than one. This alone justifies that w_t should be monotone increasing. To verify whether this remains true for delinquent loans we average period by period the wealth levels of loans about to miss their first payment. We

Figure 1: *Wealth and DSCR Prior to First Delinquency*



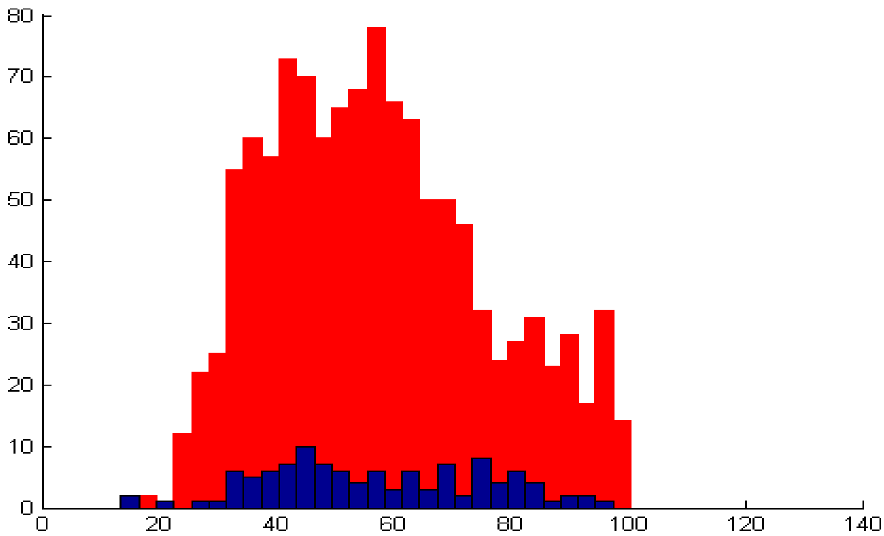
thus obtain an average wealth path leading to a first delinquency. We display this, along with the average wealth for performing loans in figure 1. On the x-axis, we measure time in month. These paths are defined as the observed wealth levels up to a first missed payment for the series Wealth First Delinquency or up to the end of the recording period for the other series. For information we have also included the contemporaneous DSCR for the loans about to miss their first payment. On the Y left axis, we keep track of the magnitude of the wealth level, whereas the Y-axis on the right measures the contemporaneous DSCR.

We observe a very clear shift in the slope of the wealth levels leading to a missed payment after period 20. This shift is aggravated about two years prior to missing their first period, where the wealth level remains nearly flat. On the other hand the wealth level for performing loans exhibits a steady increase. One intuitive explanation is to defined the area between these two curves as the wealth lost as a consequence of poor market performances. When this loss is too important, it results in a missed payment.

5.3 Property Value and Equity Levels

Property value estimates come from different sources. Appraisals at origination are taken from the data provided by the Bank of America loan level data set. These initial values are in turn updated quarterly with the quarterly National Council of Real Estate Investment Fiduciaries (NCREIF) property and income series, stratified by eight geographic regions and four property types. We finally obtain by linear interpolation, monthly property value estimates. Given that we have market performance data, we control that large swings in NOI are indeed reflected in our property value estimates. NOI's time series are cleaned of any seasonal component and monthly property value estimates are adjusted for any spike in operating income exceeding 15%.

Figure 2: *Histogram of the Equity Levels of Performing loans and Loans at time of Default*



It can naturally be argued that in theory the property value is in fact related to the income level. We argue back that while this is true in perpetuity, it need not be true for short periods of time. While this methodology for estimating the property value might reflect for the most part movements in real estate prices at the macro level and thereby miss the specificity of each property, we claim that it preserves the independence between wealth and equity. Robustness checks will however verify that parameter estimates are not severely altered by property value estimate that track more closely the operating income trajectory. We next amortize all the loans assuming that the rate remains constant over time at the level the loan was originated and derive equity level. Figure 2 exhibits equity levels of performing loans against the equity level of loans at the time of default. Surprisingly enough, it can be noticed that most of the loans in our sample default with a strong positive equity. These are almost uniformly distributed between 20 and 80 Scheduled Debt Services (SDS). In fact, the average equity level is found to average about 57.1081 SDS for loans that have defaulted and about 57.3953 SDS for performing loans. On the other hand the cumulated wealth level for loans that have defaulted averages 15.36 SDS at the time of default whereas performing loans average 34.57 SDS. This together seems to indicate that the cumulated wealth variable might be a significant factor taken into account when deciding whether or not to default.

6 Empirical Analysis

The first stage is to estimate the parameters θ of the transition probability $p(x_{t+1}/x_t, u_t, \theta)$. At this stage we do not need to find the fixed point EJ_θ and proceed with a standard parametric maximum likelihood estimation. Using the principle of conditionality, it can be easily shown that the resulting estimators are consistent and asymptotically normally distributed. Given the discretization of the wealth and equity variable proposed in (4.8) we notice that the transition

Table 1: Across Time Estimates of Transition Probabilities for Wealth Increments *(Standard errors in parentheses)*

	$t=1$	$t=10$	$t=20$	$t=30$	$t=40$	$t=50$	Full sample
θ_{w-1}	0.0106	0.0207	0.0257	0.0395	0.0375	0.0287	0.0289
	(0.0026)	(0.0037)	(0.0042)	(0.0055)	(0.0056)	(0.0060)	(0.0006)
θ_{w0}	0.1442	0.1629	0.2091	0.2285	0.2426	0.2376	0.2079
	(0.0091)	(0.0095)	(0.0109)	(0.0118)	(0.0127)	(0.0154)	(0.0015)
θ_{w1}	0.6844	0.6822	0.6081	0.5889	0.5785	0.5888	0.6137
	(0.0120)	(0.0120)	(0.0130)	(0.0138)	(0.0146)	(0.0178)	(0.0018)
θ_{w2}	0.1488	0.1195	0.1392	0.1265	0.1291	0.1332	0.1345
	(0.0092)	(0.0084)	(0.0092)	(0.0093)	(0.0099)	(0.0123)	(0.0013)
θ_{w3}	0.0120	0.0147	0.0178	0.0166	0.0122	0.0117	0.0151
	(0.0028)	(0.0031)	(0.0035)	(0.0036)	(0.0032)	(0.0039)	(0.0005)
Log Likelihood	-1,390	-1,427	-1,499	-1,400	-1,262	-824	-75,151
-2LL	2,780	2,854	2,998	2,800	2,524	1,648	150,302

probability is fully characterized with 25 parameters. Before estimating the joint distribution, we present in table 1 and 2 the marginal transition probabilities $\theta_{wi} = p(w_{t+1} = w_t - u_t + i)$ and $\theta_{ei} = p(e_{t+1} = e_t + PV(u_t) + i)$ (table 2 can be found in the appendix). In particular we want to ascertain whether these transition probabilities are stationary over time. A lack of stationarity would suggest that in order to preserve the markovian structure we would need to augment the state variable with the time dimension.

As can be seen from the bottom row of table 1 and 2, the log likelihood for the restricted model where all parameters are homogeneous across time is -75,151 whereas the unrestricted models have log likelihood that rise at least to -1,500. A test of the homogeneity is rejected with high level of confidence as the likelihood ratio -2LL $= 2(LL_{unrestricted} - LL_{restricted}) \sim \chi^2_{350}$. As the .1% critical value for a χ^2 distribution with 350 degrees of freedom is 437.48, we reject the null hypothesis that the transition probabilities are stationary. Unfortunately if we augment the state variable by adding the time dimension, we increase the computational burden by a factor 80. This alone prevents from estimating the parameters in a timely manner. Instead we decide to proceed with the homogeneous transition probabilities and as a robustness check, evaluate the model at different points in time.

We finally carry out the maximum likelihood estimation of the joint transition probability distribution of w_t and e_t. We proceed as previously but this time only show the stationary restricted estimate derived over the whole sample. Note however that we reject her again the homogeneity test and therefore will carry on the fixed point algorithm with the stationary distribution at different period over the lifetime of the loan. We use the notation introduced previously, i.e. θwi and θei. In the following table we understand the intersection of a row and a column as the joint probability that such an increment will take place given some control u_t. For instance the probability that can be found at the third row and second column, that is 0.4321 is in fact $p(\ e_{t+1} = e_t + PV(u_t) + 1\ ,\ w_{t+1} = w_t - u_t + 0)$.

Table 2: Transition probability distribution for the joint increments in equity and wealth

	θ_{w-1}	θ_{w0}	θ_{w1}	θ_{w2}	θ_{w3}
θ_{e-1}	0.0046	0.0208	0.0034	0.0001	0.0001
θ_{e0}	0.0299	0.1483	0.0276	0.0017	0.0003
θ_{e1}	0.0699	0.4321	0.1026	0.0078	0.0013
θ_{e2}	0.0169	0.0923	0.0232	0.0018	0.0003
θ_{e3}	0.0019	0.0102	0.0024	0.0005	0.0001

$\theta_{ei}=p(e_{t+1}=e_t+PV(u_t)+i)$ and $\theta_{wi}=p(w_{t+1}=w_t-u_t+i)$

6.1 Estimate of the Structural Parameters

Using the estimates of $\theta_{ei}\theta_{wj}$ from the first likelihood function as initial consistent starting values, we estimate in stage 2 the remaining structural parameters (C^{del}, C^{def}, θ_{ij}) from the cost functions (3.7), (3.8), (3.10) and (3.12) using the partial likelihood function (4.7). This estimation step requires the *inner* calculation of the fixed point for any parameters value while maximizing the log likelihood with the *outer* algorithm. This stage estimation results are presented in table 3 and 4. We have estimated the structural model $P(u/x,\theta)$ for diverse parametric specification of the cost function $g_{ij}(u)$ resulting in a number of parameter estimates that are summarized in table 3. The estimation has been made with the following parameters: The dimension in wealth is set to 40, the dimension in equity to 80, delinquency can not exceed 120 days past due, the five decisions available are to make either 0,1,2 or 3 monthly payments or default, the discount factor is set to 0.998 while the grace period τ is set to 16 months. Finally the estimation is carried with the stationary transition probability distribution.

Our first task is to select a parametric functional form that fits the data best. It is firstly interesting to note that both the wealth and delinquency variable cannot be discarded. Model A and B control whether any of these two variables is significant at all. A likelihood ratio test rejects at the 0.1% the null hypotheses that either the delinquency parameters or wealth parameters are equal to zero. In particular we can compare models A and B to the nested Model C. In both cases we find a deviance statistic that exceeds the critical value for a χ^2 with two degrees of freedom at even the most restrictive confidence levels. Another evidence of the importance of the delinquency variable is witnessed by comparing Model H and Model J or by comparing similarly Model F and Model G. in each case, these two models have the same parametric specification except that Model H and Model F restrict θ_{12} to zero. It results that Model H and F fit the data poorly and lack the predictive power of Model G or J. Moreover, we test the null hypotheses that equity is a significant variable. When comparing the two nested models I and J, the difference in between the deviance statistic for the restricted model (I) and the unrestricted model is equal 14.6, which exceeds the critical valu of a χ^2 distribution with 1 degree of freedom at any confidence level. It results that we cannot reject the null hypotheses that equity is not significant. While we can logically drop the Model A, B, F and H, the choice between the five remaining models is less obvious. Model G has undeniably the best technical criteria, but lacks the intuitive simplicity of Model C when it comes to interpret the parameters value. Our decision is therefore a compromise between a functional form with the highest likelihood value and a cost function that is both intuitively simple to assess and consistent with some of priors. These objectives lead us to select Model C over the other parametric functional

Table 3: Structural Estimates for Several Cost Function

	Model A	Model B	Model C	Model D	Model E
C^{del}		5.8783	5.8517	5.8504	5.8508
$\theta_{12} d_t$		-1.1805	-1.1622	-1.1598	-1.158
$\theta_{22} w_{t+1}$	-0.58		-0.0069	-0.0157	-0.0167
$\theta_{32} \sqrt{w_{t+1}}$	7.426			0.0867	0.0865
$\theta_{32} e_{t+1}$		0.0008	0.0011	0.001	0.0015
θ_{32} / e_{t+1}	-367.5342				1.01
C^{def}	134.3525	0.2516	1.3003	1.0971	1.0971
Goodness of fit					
Log Likelihood	-24,961	-3,568.1	-3,530.1	-3,529	-3,530.9
-2LL	49,922	7,136.2	7,060.2	7,058	7,061.8
AIC	49,930	7,144.2	7,070.2	7,070	7,075.8
n of parameters	4	4	5	6	7
Algorithm					
Iterations	63	37	14	61	13
Function Count	355	255	108	693	136
Time for convergence	6.99 hours	4.88 hours	2.44 hours	14.19 hours	3.29 hours

forms.

In reading the parameter estimates it is worthwhile noting the shape of the cost function attributed to any delinquency state. A performing loan will incur a cost of 5.8517 Scheduled Debt Service payments when becoming delinquent. This cost is by nature unobservable and corresponds to the impact perceived by the borrower for the loss of goodwill. However once delinquent, it becomes apparent that any further missed payment will instead profit the borrower by 1.1805 SDS. This clearly suggest that there is a strong resistance to a first missed payment, but once the loan is 30 days delinquent, the borrower has a stronger incentive to profit from the additional cash freed for each payment delayed. Similarly, the parameters suggest that every additional unit of wealth will profit the borrower by 0.69% a month. On an annual bases this adds up to about 8.6%. This seems to be close to the average cap rate for a full service or suite hotel. Another surprising outcome of the estimation procedure is the magnitude of the unobservable cost to default. The calibration seems to suggests that the additional cost to default amounts to about a quarter of the original cost for becoming delinquent. In this context, the model suggests that borrowers apprehend more missing their first payment than actually defaulting. We pounder how much of that magnitude is the consequence of the grace period chosen or the transition probability. Meaningful estimates of the unobservable cost of default will be derived for different loan ages and grace periods. The last parameter estimate is somewhat counter-intuitive. The equity component hints that for any appreciation of her equity, the borrower will incur a 1.32% cost on annual bases. While this could be just the consequence of a lack of precision in estimating the property value, we also consider the possibility that

Table 4: Structural Estimates for Several Cost Function (part II)

	Model F	Model G	Model H	Model I	Model J
C^{del}	4.0564	5.8401	4.0849	5.8876	5.8519
$\theta_{12} d_t$		-1.1578		-1.1697	-1.1614
$\theta_{22} w_{t+1}$	-0.0208	-0.0158	-0.0043	-0.0080	-0.0081
$\theta_{32}\sqrt{w_{t+1}}$	0.1587	0.0876			
θ_{32}/w_{t+1}			-0.2399	-0.5832	-0.5625
$\theta_{32} e_{t+1}$			0.0026		0.001
θ_{32}/e_{t+1}	-8.8535	-3.121			
C^{def}	-1.7693	1.8733	-3.9491	1.2194	1.3003
Goodness of fit					
Log Likelihood	-4,320.70	-3,527.30	-4,328.20	-3,536.3	-3,529.00
-2LL	8,641.40	7,054.60	8,656.40	7,072.6	7,058.00
AIC	8,651.40	7,066.60	8,666.40	7,082.6	7,070.00
n of parameters	5	6	5	5	6
Algorithm					
Iterations	70	81	55	41	71
Function Count	546	686	588	336	805
Time for convergence	11.23 hours	14.02 hours	11.36 hours	6.74 hours	16.18 hours

something more profound has surfaced. Whereas most, if not all, reduced form models derive LTV parameters such that any increase in equity will result in a decrease in the likelihood to default, we remind the reader that our model is assessing the long term effect of an immediate change in equity. Reduced form models, such as hazard models, are driven by intensities in the underlying factors. Our structured model is on the other hand assessing the expected cost in perpetuity of an immediate change. While we can conjecture that more precise measures of the property value could further confirm or rectify this result, we also accept the idea that default of commercial loans is in fact largely triggered by low performance results and less a function of the equity at stake. If we understand the initial equity as a sunk cost, then rational borrowers should make their decision to walk away based mostly on economical performance, that is the expected propensity of the property to generate wealth, and less on the initial investment that was necessary to enter the market.

6.2 Conditional Probabilities

One of the fundamental outcomes of the model is that in order to derive the log likelihood, we had to derive for every data point the conditional probability to observe the decision $\{u_t\}$ given the state $\{w_t, e_t, d_{t-1}\}$ in which the loan was. Now that the we have calibrated the model with its optimal cost parameters, we can in turn use these optimal conditional probabilities to assess whether the model has any predictive power. Figure 3 to 5 exhibit the conditional probability that a performing loan will either remain performing, miss a payment or default while figure 6 to 9 present the conditional probability that a 60 days delinquent loan will either become 90 days delinquent, remain 60 days delinquent, cure or default. These conditional probabilities are derived over all equity and wealth levels. They were derived from (4.5) and as a consequence, for any given state $\{w_t, e_t, d_{t-1}\}$, sum to one when considering all possible choices, i.e. $u_t \in U\big(\{w_t, e_t, d_{t-1}\}\big)$, available to the borrower in that state.

Figure 3 shows more specifically the conditional probability that a performing loan will remain performing. The probabilitties range from almost 1 to a little more than 0.97. The model suggests that this probability decreases as wealth decrease and equity increase. The most noticeable effect is the cumulated impact of a decrease in wealth. Also as wealth drops to a level of less than five SDS, we notice a severe drop in the probability that a performing loan will make its scheduled debt service payment. It is as if, below a given level a wealth, distress costs had exceeded the cost for becoming delinquent and that as result, the borrower preferred to miss her payment rather than incur further distress costs. These distress cost can be, but are not limited to, a loss of a credit line established with her providers or the inability to face the cost from operations, such as wages.

On the other hand Figure 4 presents the conditional probability that a performing loan will miss it's next payment without defaulting. In this case the probabilities range from 0.75% to a little above 1.05%. Note that missing a payment is not binding per say as long as it is not triggered by a default decision. A missed payment can be cured and can be perceived as a financing decision when making the debt service payment is not the optimal allocation of the borrower's wealth. The graph exhibits a spike in this probability around a wealth level of SDS. Below that level the probability of just missing a payment without defaulting drops significantly. On the other side, the probability diminishes smoothly as the wealth level increases as it could be expected. Equity seems to have in general very little impact on this decision. Naturally a missed payment can also be the consequence of a decision to default. In figure 5 we detail this conditional probability and notice that most default decisions will take place as the wealth level has dropped below the level of 5 SDS described above. These probability can be as high as almost 2.5% for very low levels of wealth and, surprisingly, high levels of equity. This positive correspondence between equity and the probability to default is naturally the consequence of the parameter estimate on the equity factor. It is nonetheless surprising, as one would instead intuitively prefigure that high levels of equity offer a bigger loss in the event of a default.

We investigate next the case of 60 days delinquent loans. The conditional probabilities displayed in figure 6 to 9 are interesting as they outline two very significant outcomes. Firstly that a 60 days delinquent loan will most likely become 90 days delinquent without necessarily defaulting. In fact the probability of a missed payment without default can be as high as 97% for very low levels of wealth and only amount to 1.3%, in the worst case, as a consequence of default. Secondly, the model captures rightly the incidence of wealth on the probability to cure the delinquency. As wealth rises the probability to make up for all late payments can reach 60%. Of course one might wonder why a loan with a high level of wealth could have chosen in the first place to miss a payment. It has however brought to our attention that delinquency can also be used to obtain the undevised servicer's attention in the case of, for example, refinancing.

Table 5: Across Time Estimates of Transition Probabilities for Equity Increment *(Standard errors in parentheses)*

	$t = 1$	$t = 10$	$t = 20$	$t = 30$	$t = 40$	$t = 50$	Full sample
θ_{e-1}	0.0485	0.0708	0.1642	0.2198	0.2086	0.0953	0.1231
	(0.0055)	(0.0066)	(0.0099)	(0.0116)	(0.0120)	(0.0106)	(0.0012)
θ_{e0}	0.6494	0.7043	0.6867	0.6830	0.6850	0.7141	0.7037
	(0.0123)	(0.0118)	(0.0124)	(0.0131)	(0.0137)	(0.0163)	(0.0017)
θ_{e1}	0.2822	0.2023	0.1342	0.0798	0.1047	0.1828	0.1592
	(0.0116)	(0.0104)	(0.0091)	(0.0076)	(0.0090)	(0.0140)	(0.0014)
θ_{e2}	0.0166	0.0214	0.0128	0.0119	0.0017	0.0078	0.0119
	(0.0033)	(0.0037)	(0.0030)	(0.0030)	(0.0012)	(0.0032)	(0.0004)
θ_{e3}	0.0033	0.0013	0.0021	0.0055	0	0	0.0020
	(0.0015)	(0.0009)	(0.0012)	(0.0021)	(0)	(0)	(0.0002)
Log Likelihood	-1312	-1271	-1252	-1109			-61104
-2LL	2624	2542	2504	2218			1222208

Figure 3: *Conditional Probability that a performing loan will make it's next scheduled payment*

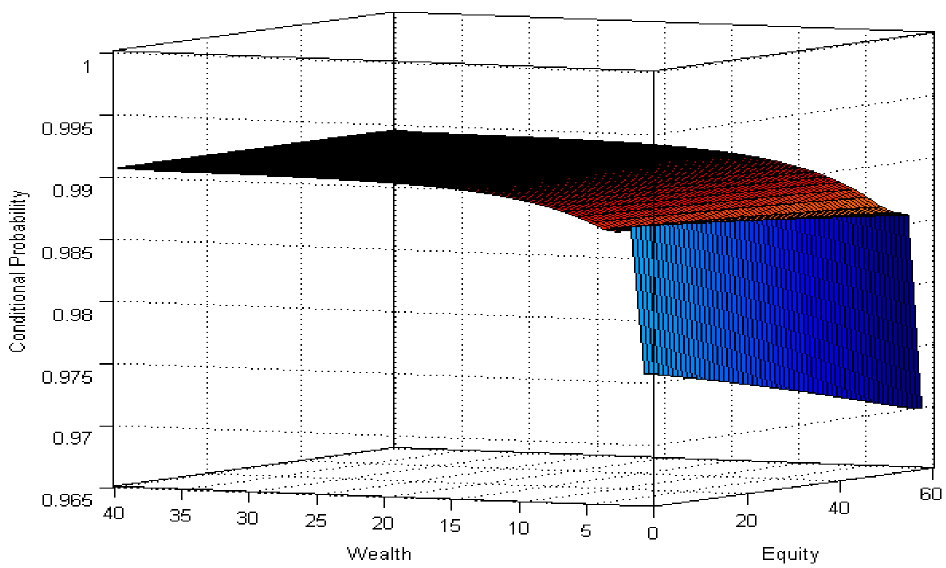


Figure 4: *Conditional Probability that a performing loan will miss it's next scheduled payment*

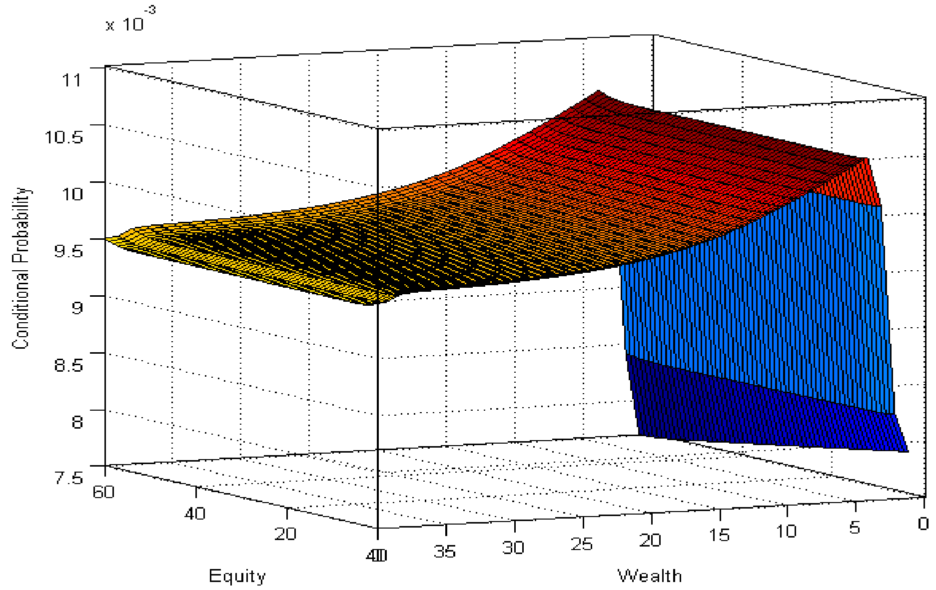


Figure 5: *Conditional Probability that a performing loan will default*

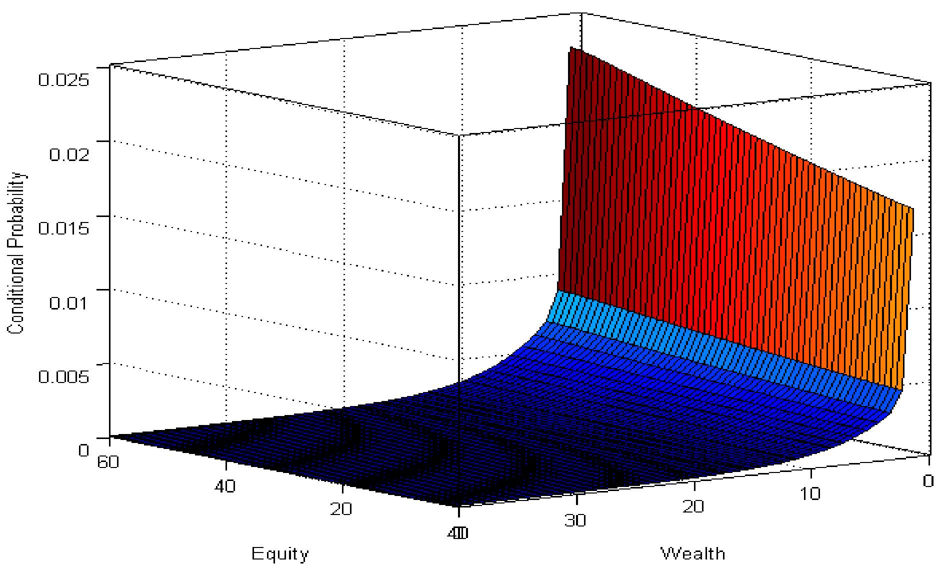


Figure 6: *Conditional Probability that a 60 days delinquent loan will become 90 days delinquent*

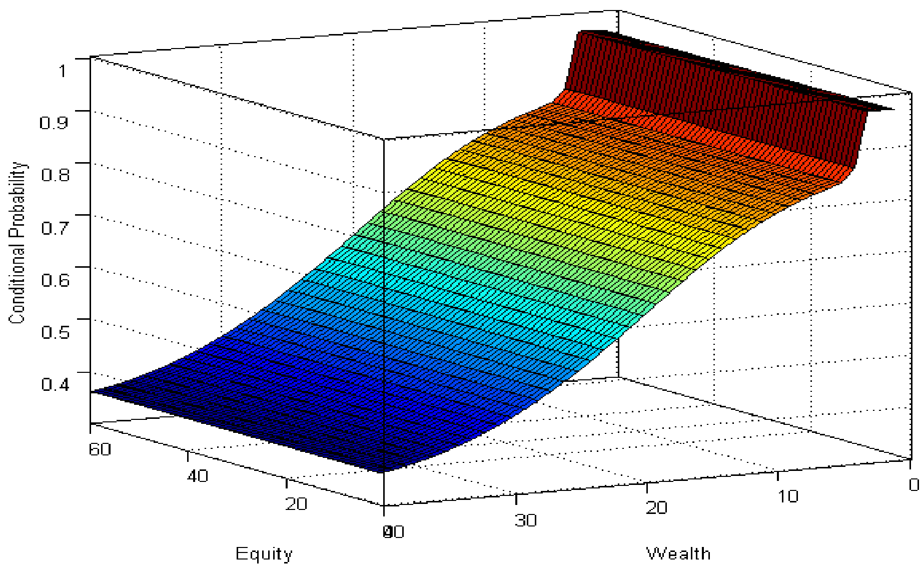


Figure 7: *Conditional Probability that a 60 days delinquent loan will remain 60 days delinquent*

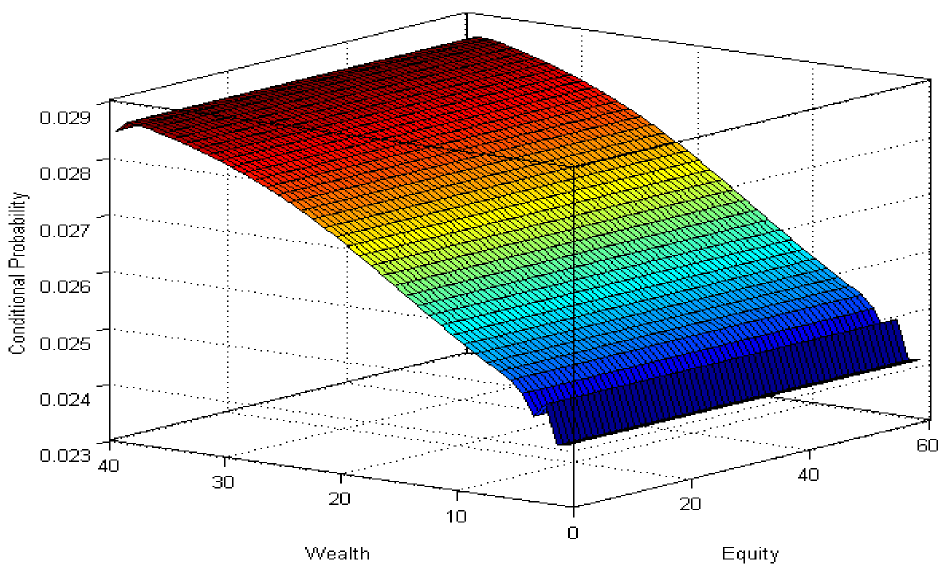


Figure 8: *Conditional Probability that a 60 days delinquent loan will cure*

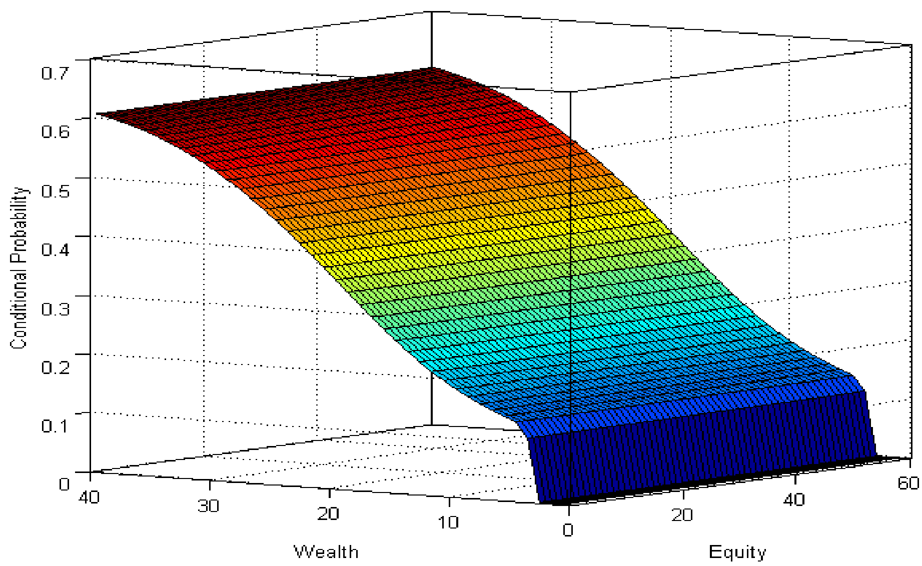
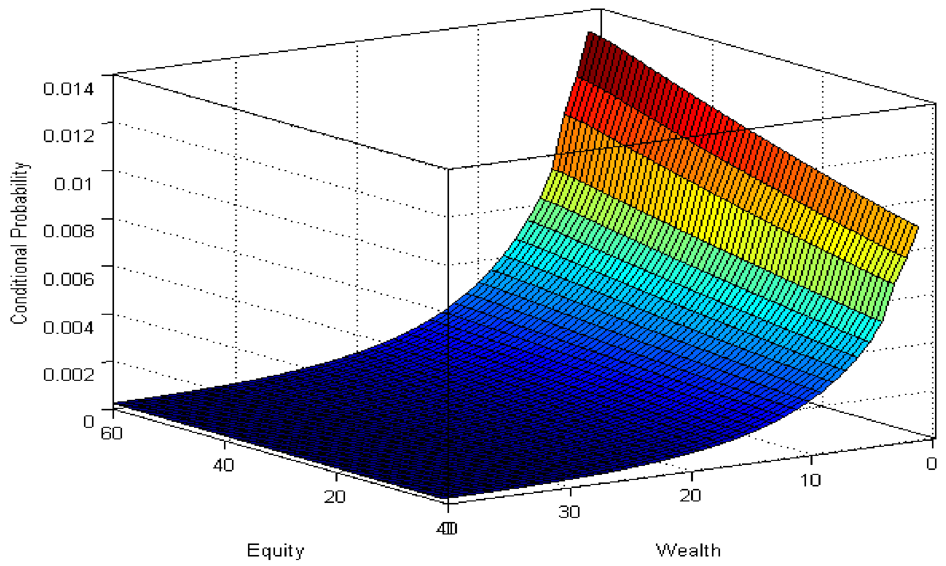


Figure 9: *Conditional Probability that a 60 days delinquent loan will default*



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